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USE OF ASYMPTOTIC METHODS FOR CALCULATION OF A TURBULENT BOUNDARY LAYER

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Using the method of joined asymptotic expansions and the two-layer scheme of incompressible-fluid flow in a turbulent two-dimensional boundary layer, we have derived relations for calculating the coefficient of turbulent friction and the distribution of the Reynolds stress over the cross sections of the layer in the first asymptotic approximation. It is shown that in the zone of a defect on bodies having a relatively large disturbance-surface curvature the velocities should be separated into vortex velocities, which are due to the coherent structures, and potential velocities, caused by the transverse pressure gradient. From the available experimental data on the structure of the flow we inferred that the redundant-velocity profile obtained in the limiting asymptotic approximation, which, in this case, as in the case of self-similar (equilibrium) regime of flow, is locally dependent on only the Clauser parameter, is universal.

Introduction. The calculation of a flow in a turbulent boundary layer based on the Reynolds equations is usually performed with the use of relations analogous to the laminar-boundary-layer equations. The only difference between them is the replacement of the molecular viscosity and the Prandtl number by the corresponding parameters accounting for the turbulence effect.

However, as experimental investigations of the structure of near-wall turbulent flows have shown, such "diffusion" approaches to the closure of boundary-layer equations are essentially incorrect. Mass, momentum, and heat transfers across a turbulent boundary layer are for the most part due to the coherent "pin-like" vortex structures [1, 2]. To put it otherwise, the mechanism of heat and mass transfer is similar to the mechanism of convective transfer.

The asymptotic approach suggests that, when the coefficient of surface friction approaches zero, the disturbances in the main external part of a layer are small [3]; therefore, the transverse pressure difference cannot be ignored in the general case. In the case of convective momentum transfer, in the limiting asymptotic approximation the redundant-velocity profile should depend on only the place (local) characteristics of the flow and the conditions on the walls.

The present work is devoted to substantiation of the above-mentioned features of an incompressible fluid flow in a turbulent boundary layer and investigation of the possibility of their use.

Estimation of the Order of Magnitude of the Parameters of a Turbulent Two-Dimensional Boundary Layer. Let us assume that all the linear dimensions are related to the characteristics longitudinal dimension of the body. In accordance with the two-layer scheme of flow used in the present work, we introduce the characteristic thickness of the boundary layer δ and the characteristic transverse dimension of the near-wall zone of the flow h. The solution is sought in the asymptotic form: $\delta \to 0$ at a small parameter $\varepsilon \to 0$ and a constant pressure distribution. The parameter ε is determined as

$$\varepsilon = \sqrt{\frac{\tau_w}{\rho u_0^2}} = \sqrt{\frac{c_f}{2}} \,. \tag{1}$$

It is shown in [3] that, for a plane-parallel flow in a channel with a transverse dimension R, the velocity-defect law is true in the external zone of the flow of the order of $O(R)$ at a friction coefficient approaching zero. To put

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it otherwise, the velocity is written as a binomial asymptotic expansion where the first term is of the order of $u_0 = u_0(x)$ and the second term has a relative order ε.

A flow in a turbulent boundary layer with $\delta \to 0$ approaches to a layered flow; in this case, the velocity-defect law should be true in the first approximation. Therefore, the ratio h/δ will be assumed, as in [3], to be exponentially small in comparison with ε:

$$h/\delta = O\,[\exp\,(-1/\varepsilon)]\,. \tag{2}$$

Then the contribution of the near-wall layer to the displacement thickness δ^* and the momentum thickness of the boundary layer θ can be neglected with an exponentially small error in calculating these parameters.

Contrary to the case of flow in a channel, the transverse dimension of the boundary layer δ is not known in advance and represents a fairly conditional quantity. Let us relate the quantity δ to the displacement thickness and, using the estimation following from the velocity-defect law $1 - u/u_0 = O(\varepsilon)$, obtain, as in [4]:

$$\delta^* = \delta \int_0^\infty (1 - u/u_0)\, d\left(\frac{y}{\delta}\right) = O\,(\varepsilon\delta)\,, \quad \delta = O\left(\frac{\delta^*}{\varepsilon}\right). \tag{3}$$

The order of magnitude of the quantity δ will be determined using the integral equation momentum conservation

$$\frac{d\theta}{dx} = O\left(\frac{d\delta^*}{dx}\right) = O\,(c_{\mathrm{f}}) = O\,(\varepsilon^2)\,. \tag{4}$$

It hence follows that

$$\delta = O\left(\frac{\delta^*}{\varepsilon}\right) = O\,(\varepsilon)\,, \quad \frac{\delta^*}{\varepsilon^2} = \gamma = O\,(1)\,. \tag{5}$$

The estimations obtained will be used in determining the asymptotic error of the turbulent-boundary-layer equations and deriving the corresponding relations.

Asymptotic Error of the Turbulent-Boundary-Layer Equations. Relations for the Velocity-Defect Region. According to the data of [3], an asymptotic solution of the problem on a layered flow in a channel has an exponentially small error corresponding to estimation (2). A solution in the near-wall zone of a turbulent layer, where the flow approaches to a layered flow with the above-indicated accuracy, should have an equally small error. The flow in the external zone of the turbulent layer differs from the layered flow much more substantially. Let us write the Reynolds equation for this zone of the flow, taking into account the fact that the viscosity effect can be ignored. In the curvilinear, orthogonal coordinate system related to the surface of the body, from the nonstationary Euler equations [5] we obtain

$$\begin{gathered} u\,\frac{\partial u}{\partial x} + v\,\frac{\partial qu}{\partial y} = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) + \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial q\tau_{xy}}{\partial y} + \frac{\tau_{xy}}{r}\right)\frac{1}{\rho}\,, \\ u\,\frac{\partial v}{\partial x} + v\,\frac{\partial qv}{\partial y} - \frac{u^2 + v^2}{r} = -\,q\,\frac{\partial}{\partial y}\left(\frac{p}{\rho}\right) + \left(\frac{\partial q\tau_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} - \frac{\tau_{xx}}{r}\right)\frac{1}{\rho}\,, \\ \frac{\partial u}{\partial x} + \frac{\partial qv}{\partial y} = 0\,. \end{gathered} \tag{6}$$

Here, $q = 1 + y/r$; $r(x)$ is the local radius of the body-surface curvature ($r > 0$ for a convex surface).

Let us estimate the order of magnitude of the terms in (6), using (5) and the velocity-defect law: $x = O(1)$, $y = O(\delta)$, $u/u_0 = 1 + O(\varepsilon)$, $q = 1 + O(\varepsilon/r)$ ($r = O(1)$ in the general case). From the continuity equations follows $v/u_0 = O(\varepsilon)$. According to the experimental data and the condition $\tau_w = O(\tau_{xy}) = \rho u_0^2 O(\varepsilon^2)$, $(\tau_{xx}, \tau_{xy}, \tau_{yy}/\rho u_0^2 = O(\varepsilon^2)$.

Based on the estimations obtained, we drop the term of the relative order of $O(\varepsilon^2)$ in (6), assuming that the pressure on the surface of the body $p_w(x)$ is preassigned:

$$u\frac{\partial u}{\partial x} + v\frac{\partial qu}{\partial y} = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) + \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y},$$

$$\frac{u^2}{r} = \frac{\partial}{\partial y}\left(\frac{p}{\rho}\right) \quad \text{or} \quad \frac{p}{\rho} = \frac{p_w(x)}{\rho} + \frac{1}{r}\int_0^y u^2 dy, \tag{7}$$

$$\frac{\partial u}{\partial x} + \frac{\partial qv}{\partial y} = 0.$$

The traditional equations of a turbulent boundary layer follow from (7) on condition that $q = 1$ and $p/\rho = p_w/\rho$. This (at the same error $O(\varepsilon^2)$) calls for the condition $r \ge O(1/\varepsilon)$, which corresponds bodies with a thickness of the same order or smaller than the thickness of the boundary layer. Only for such bodies can the velocity defect of the relative order of ε be calculated using classical equations. In the case where $r = O(1)$, it is necessary to take into account the pressure difference along the normal to the boundary layer (as already noted in [6]). It should be noted that, in both cases, the relative error in determining the velocity is equal to $O(\varepsilon^2)$. Thus, it is incorrect to seek the next asymptotic approximation, i.e., to deduct the velocity expansion terms of the order of $O(\varepsilon^2)$ in the case where classical turbulent-boundary-layer equations are used (i.e., in the case where the stresses (τ_{xx}, $\tau_{yy} = O(\varepsilon^2)\ \rho u_0^2$) are ignored).

Hence it follows that the asymptotic solution (7) can be sought by linearization on the assumption that the disturbances caused by the transverse pressure gradient are independent of the other disturbances. Taking into account the aforesaid, we write

$$u = u_0(x)\,[1 + \varepsilon F(x, \eta) + \varepsilon f(x, \eta) + \varepsilon^2 E(x, \eta) + O(\varepsilon^3)], \quad v = u_0(x)\,[\varepsilon\varphi(x, \eta) + O(\varepsilon^2)]. \tag{8}$$

Here, $\eta = y/\varepsilon = O(1)$; εF is the relative potential velocity disturbance caused by $\partial p/\partial y$, and εf is the vortex disturbance. In the main approximation, substitution of (8) into (7) gives the Bernoulli equation: $u_0^2/2 + p_w/\rho = \text{const}$.

The pressure distribution is described as

$$\frac{p}{\rho} = \frac{p_w}{\rho} + \varepsilon\eta\frac{u_0^2}{r} + O(u_0^2\varepsilon^3). \tag{9}$$

From the continuity equation follows

$$\varphi = -\eta\frac{d\ln u_0}{dx}. \tag{10}$$

For a potential flow about a body (the Bernoulli constant does not change across the streamlines), from (9) and the Bernoulli equation follows

$$F = -\frac{\eta}{r}. \tag{11}$$

In the linear approximation considered, the vortex disturbances f depend on only $u_0(x)$ and τ_{xy}. For them, from the first equation of (7) and Eq. (10) we obtain

$$\frac{\partial f}{\partial x} + \frac{d\ln u_0}{dx}\left(2f - \eta\frac{\partial f}{\partial \eta}\right) + f\frac{d\ln\varepsilon}{dx} = \frac{\partial T}{\partial \eta}, \tag{12}$$

where $T = \tau_{xy}/(\rho u_0^2 \varepsilon^2) = O(1)$.

Asymptotic Joining of Solutions in the External and Internal Zones of a Flow. Joining of solutions will be performed, using the data of [3], at any degree of the surface roughness. It should be noted that, according to (2) and (11), in the region of joining the contribution of the potential disturbances to the solution is exponentially small. Therefore, only the function f, for which it is necessary to find the boundary conditions at $\eta \to 0$, should take part in the indicated procedure.

The solution in the near-wall zone of the flow, determined by the conditions on the surface and by the shear stresses invariable along the normal to the body, should be practically identical to the solution for the flow in the channel, the only difference being the parametric dependence on x. Because of this, all the data of [3] concerning the joining of solutions can be used in the case of a turbulent boundary layer if the characteristic transverse dimension of the channel R is replaced by the thickness of the layer δ^*/ε and the ratio of the dynamic velocity to the characteristic velocity u_0 ($\sqrt{\lambda/8}$ in [3]) is replaced by ε. As a result of this replacement, we obtain the following boundary condition for Eq. (12):

$$f \to A \ln (ay/R) = A \ln (a\eta/\gamma) \quad \text{at} \quad \eta \to 0 . \tag{13}$$

The relation for determining the small parameter ε can be written in a similar manner:

$$\frac{1}{\varepsilon} = A \ln \frac{\delta^*}{a\varepsilon h} = A \ln \frac{\gamma\varepsilon}{ah} . \tag{14}$$

The result obtained corresponds to the conclusions made in [7], where it has been shown that the solutions in the internal and external regions of the flow can be directly joined without the introduction of a third intermediate zone.

The parameters γ, a, and h in (13) and (14) are dependent on x in the general case. Let us assume that these dependences are fairly smooth or $d \ln \gamma/dx$, $d \ln a/dx$, and $d \ln h/dx = O(1)$. Then, from (14) follows

$$\frac{d \ln \varepsilon}{dx} = O(\varepsilon) . \tag{15}$$

Thus, the corresponding term in (12) can be ignored, which does not introduce an additional asymptotic error into the solution. Using (8), we introduce the displacement thickness by the formula

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_0} + \varepsilon F\right) dy = -\varepsilon^2 \int_0^\infty f d\eta \, (1 + O(\varepsilon)) . \tag{16}$$

In the case where δ^* is determined in this manner, only the displacement action of the vortex disturbances is taken into account, since $\delta^* \equiv 0$ in their absence (in the absence of a boundary layer). Since $f = 0$ outside the layer, the integral of f should converge.

Let us integrate (12) over η from 0 to ∞. Taking into account (15) and the conditions $T(0) = 1$ and $T(\infty) = 0$, we obtain

$$\frac{d\gamma}{dx} + 3\gamma \frac{d \ln u_0}{dx} = 1 , \quad \gamma = \frac{\delta^*}{\varepsilon^2} = O(1) . \tag{17}$$

Expression (17) is an integral momentum-conservation equation, in which the terms with a relative order $O(\varepsilon)$ are dropped and the velocity u_0 is not the velocity at the outer boundary of the layer in the general case. In this case, the second term in (17) involves the known Clauser parameter β:

$$\beta = -\gamma \frac{d \ln u_0}{dx} . \tag{18}$$

The linear equation (17) has a solution

$$\gamma = \left(C + \int_{x_1}^{x} u_0^3 dx \right) u_0^{-3} . \tag{19}$$

The constant $C = \gamma_1/u_{01}^2$, where $\gamma_1 = \gamma$ and $u_{01} = u_0$ at $x = x_1$ corresponding to the "beginning" of the turbulent boundary layer, can be formally taken as zero at $x_1 = 0$ on the assumption that the initial conditions have a small effect on the turbulent layer at large values of x.

Let us drop the term with $d \ln \varepsilon/dx$ and pass to the variable $\zeta = \eta/\gamma$ in Eq. (12) or, put otherwise, to the variable y normalized to the thickness of the boundary layer δ^*/ε.

Then, using (17) and (18), we obtain

$$\gamma \frac{\partial f}{\partial x} - 2\beta f - (1 + 2\beta)\, \zeta \frac{\partial f}{\partial \zeta} = \frac{\partial T}{\partial \zeta} , \quad \zeta = \frac{\eta}{\gamma} . \tag{20}$$

The boundary conditions are as follows: $f \to 0$, $T \to 0$ when $\zeta \to \infty$; $f \to A \ln \zeta$;, $T \to 1$ when $\zeta \to 0$.

The self-similar (or equilibrium, according to the Clauser terminology) solution of this equation independent of x is possible only at β = const, which is the necessary condition for the existence of this solution. Then, assuming that $C = 0$ and $x_1 = 0$ in (19), from (17), at β = const, we obtain γ = const x, $d \ln u_0/dx$ = const/x, and consequently

$$u_0 = D x^m , \quad \beta = - m\, (1 + 3m)^{-1} , \quad \gamma = (1 + 3\beta)\, x . \tag{21}$$

A self-similar flow about a plate ($\beta = 0$) was obtained and investigated in many experimental works. In the experiments [4] performed on a plane surface, self-similar flows with a positive pressure gradient ($m < 0$, $\beta > 0$) were also obtained. It should be noted that the velocities of self-similar flows realized at fairly large positive pressure gradients obey not the logarithmic law at $y \to 0$ but the law $u \approx \sqrt{y}$ [8], which directly follows from the dimensionality theory. Unlike the limiting transition $\varepsilon \to 0$, β = const used in the present work, this scheme corresponds to the transition $\varepsilon^2\beta$ = const, $\beta \to \infty$, where the velocity profile approaches to the velocity profile of a separation flow [8].

The self-similar form of (20) ($\partial f/\partial x = 0$) was obtained in [19] on the assumption that $\partial p/\partial y = 0$, i.e., only the vortex disturbances were essentially considered.

Methods of Solving the Asymptotic Turbulent-Boundary-Layer Equations. As the above-described asymptotic analysis has shown, vortex disturbances independent of the transverse pressure gradient can be investigated using classical boundary-layer equations. Because of this, the data obtained with these equations are true only for vortex disturbances and u_0 is considered as the velocity at the "outer boundary of the layer." In this case, the second approximation considered in [9] is incorrect because it is beyond the scope of the asymptotic errors of the boundary-layer equations.

It should also be noted that it is needless to use any closure hypothesis in deriving equations of the type of (20) (as has been done for the particular case where $\beta = 0$). However, the use of any closure methods makes it possible to solve (20), i.e., to find $f = f(x, \zeta)$. Clearly, having determined experimentally the profiles $f(x, \zeta)$, we can solve the inverse problem, i.e., to determine $T = T(x, \zeta)$. This approach calls for the performance of measurements for each particular case of flow and so cannot be considered as universal.

We propose some ideas based on the properties of a turbulent boundary layer, which, in the first asymptotic approximation, make it possible to use, instead of arbitrary closure hypotheses, the universality of the function

$$f\,(x, \zeta) = f\,[\beta\,(x), \zeta] . \tag{22}$$

They are supported first of all by the known Coles law of the wake in a turbulent boundary layer [11] obtained as a result of processing a large number of experimental data non-self-similar and non-self-similar flows. According to [11], the profiles f are approximated by the one-parameter dependence $f = f(\Pi, \zeta)$. Consequently, relation (22) will be true if $\Pi = \Pi(\beta)$.

The second argument in favor of our ideas is the unstable structure of the boundary layer. It is shown in [1, 2] that at fairly large Reynolds numbers a boundary layer consists of isolated aggregations (bulges) of pin-like vortex structures. These structures arise in the near-wall layer and rise to the external part of the layer due to the self-induced vertical velocity, where they take a tilting of 45–50^o with respect to the surface. The structures, extending and moving with a velocity of $(0.6–0.8)u_0$, reach the upper boundary of the layer, after which they are disrupted by a relatively powerful high-velocity liquid flow bursting from the external region of the flow to the surface. The aggregations of such structures have a transverse dimension of the order of the boundary-layer thickness.

Thus, the momentum exchange between the near-wall and external regions is largely performed in the convective way along a small longitudinal length of the order of δ. Due to this circumstance the turbulent layer differs substantially from the laminar layer. In a laminar layer, this momentum transfer is due to the molecular diffusion and occurs along lengths of the order of the body length, which corresponds to parabolic laminar-boundary layer equations.

The aforesaid allows the conclusion that all the hypotheses of the closure of turbulent-boundary layer equations, leading to equations of the parabolic type, are in contradiction with the real structure of the layer, even though they provide an acceptable accuracy for many cases of flows [12, 13].

Let us assume that the characteristic length, along which the velocity profile in a turbulent boundary layer is formed, is $\delta^*/\varepsilon = O(\delta)$. Then in this layer the flow can be considered, with a relative error $O(\varepsilon)$, as layered and a velocity-defect law similar to that in tubes and channels [3] should be true in the accepted approximation. A distinction is that this law is local, i.e., it depends on the local conditions at the outer boundary of the layer.

These conditions should account for the external-turbulence effect and the variability of the velocity u_0 in the general case. Let us assume that the first factor is absent and consider the case of small external turbulence. We will also assume that the variability of u_0 can be characterized by the derivatives of u_0 with respect to x, expressed in dimensionless form with the use of the characteristic length δ^*/ε and the dynamic velocity εu_0. Then, for the derivative of the nth order we will have

$$\beta_n = -\left(\frac{\delta^*}{\varepsilon}\right)^n \frac{1}{\varepsilon u_0} \frac{d^n u_0}{dx^n} .$$

Let us exclude the critical points of the flow where $u_0 = 0$ and regions having features in the pressure distribution (separation region, corner points). Then, assuming that $(d^n u_0/dx^n)/u_0 = O(1)$ and $\gamma = \delta^*/\varepsilon^2 = O(1)$, we obtain

$$\beta_1 = \beta = -\gamma \frac{d \ln u_0}{dx} , \quad \beta_n = O\,(\varepsilon^{n-1}) .$$

Ignoring the influence of the parameters β_n, which are asymptotically small at $n > 1$, we arrive at the conclusion that, in the general case, the local profiles of the redundant velocity should depend on only the local value of the Clauser parameter.

To put it otherwise, the function (22) f should be locally self-similar at $\varepsilon \to 0$.

It should be noted that the locally self-similar velocity profiles were used earlier in the calculation of a laminar boundary layer [14]. However, in this case, the asymptotic error is equal to $O(1)$. For a turbulent boundary layer, the local self-similarity is true to within the error $O(\varepsilon)$.

Using (22), we write out relations of the first approximation necessary for calculation of the turbulent boundary layer (formulas (14), (18)–(20)):

$$\gamma \frac{\partial f}{\partial \beta} \frac{d\beta}{dx} - 2\beta f - (1 + 2\beta)\, \zeta \frac{\partial f}{\partial \zeta} = \frac{\partial T}{\partial \zeta} , \quad f = f\,(\beta, \zeta) ; \tag{23}$$

$$f \to 0 , \quad T \to 0 \quad \text{at} \quad \zeta \to \infty ; \quad f \to A \ln a\zeta , \quad T \to 1 \quad \text{at} \quad \zeta \to 0 ;$$

$$a = a\,(\beta) , \quad \beta = -\gamma \frac{d \ln u_0}{dx} , \quad \gamma = \left(C + \int u_0^3 dx\right) u_0^{-3} , \quad \frac{1}{\varepsilon} = A \ln \frac{\gamma\varepsilon}{ah} . \tag{24}$$

To these relations we add a formula from [3] for calculating the characteristic dimension of the near;wall zone h:

$$h = 0.0331 k_e \, [\exp(-12\alpha / k_e^+) + 3.17 / k_e^+] \, . \tag{25}$$

The empirical function $\omega(\alpha)$ ($\omega(0) = 0$, $\omega(1) = 1$) involved in the formula from [3] is replaced by $\omega(\alpha) = \alpha$ in (25).

A comparison with the experimental data shows that relation (25) can be used at least in the cases where $\alpha = 1$ (sand roughness) and $\alpha = 0$ (technical roughness). Moreover, it is also true for two limiting cases where $k_e^+ \to 0$ (smooth wall) and $k_e^+ \to \infty$ (absolute roughness):

$$h = 0.105 \nu / \varepsilon u_0 \quad (k_e^+ = 0) \, , \quad h = 0.0331 k_e \quad (k_e^+ = \infty) \, . \tag{26}$$

Relations (23) are universal in that they are independent of the gas viscosity and the roughness characteristics.

With the experimentally determined $f = f(\beta, \zeta)$ (and, consequently, $a = a(\beta)$), one can obtain, using (23), the distribution of the dimensionless Reynolds pressure $T = \tau_{xy}/\tau_w$ across the boundary layer (it follows from (23) that, unlike f, the T profiles will be locally similar only at $d\beta/dx = 0$, i.e., only for self-similar flows). In the asymptotic limit $\varepsilon \to 0$, this solution should give a more reliable result, since it is not related to the fairly arbitrary closure hypotheses.

In the case where only the friction coefficient $c_f = 2\varepsilon^2$ is calculated, the problem is even more simplified, since, to do this, it is necessary to have only the experimentally determined function $a = a(\beta)$ and the initial value of γ_1 (or $\delta_1^* = \gamma_1 \varepsilon_1^2$) at $x = x_1$ and $u_0 = u_{01}$. Then, ε is calculated using (24) and (25) with a relative asymptotic error $O(\varepsilon^2) = O(c_f)$.

We now consider some particular examples of the solution of (24). Using the first relation of (26) for the case of a smooth wall, we obtain

$$\frac{1}{\varepsilon} = A \ln \frac{\mathrm{Re}_*}{0.105 a(\beta)} \, , \quad \mathrm{Re}_* = \frac{u_0 \delta^*}{\nu} \, , \quad A = 2.44 \, . \tag{27}$$

For self-similar flows, according to (21), $\gamma \sim x$ and there exists an analytical dependence of ε on Re_x:

$$\frac{1}{\varepsilon} = 2.44 \ln \frac{(1 + 3\beta) \mathrm{Re}_x \varepsilon^2}{0.105 a(\beta)} \, , \quad \mathrm{Re}_x = \frac{u_0 x}{\nu} \, . \tag{28}$$

In the case of flow about a plate [10] (u_0 = const), $\beta = 0$, $\alpha(0) = 1.27$, and consequently

$$\frac{1}{\varepsilon} = 2.44 \ln (7.5 \, \mathrm{Re}_x \varepsilon^2) \, . \tag{29}$$

An analogous formula obtained in [10] can be written in the form

$$\frac{1}{\varepsilon} = 2.44 \ln (6.1 \, \mathrm{Re}_x \varepsilon^2) \, .$$

A comparison of the results of calculation by this formula with the experimental data obtained in [10] shows that at $\mathrm{Re}_x = 10^6$–10^7 the calculation gives a slightly overstated (by approximately 1–1.5%) value of ε. Therefore, formula (29), giving a smaller (by approximately 2%) value of ε should also fairly exactly describe the experimental results.

It is assumed that $C = 0$ and $x_1 = 0$ in relations (28) and (29), i.e., the whole flow about the body is turbulent. Clearly, in the region where the regime of flow is laminar, these relations cannot be used. One would expect a relatively small accuracy of (29) at $\mathrm{Re}_x \approx 10^6$ too, i.e., in the region adjacent to the transition zone, since, in this zone, γ should be calculated, strictly speaking, using (19) at $C \neq 0$. Therefore, in the case of flow about an arbitrary body, the assumption that $C = 0$ and $x_1 = 0$ can lead to a larger error as compared to the case of flow about a plate.

We agree with the authors of [10], who argue that only the first asymptotic approximation (considered in the present work) is true within the framework of the boundary-layer approximation. In this approximation the values of

$\delta^* = \theta = O(\varepsilon\delta)$ and differ by the extra-order terms $O(\varepsilon^2\delta)$. However, the conclusion that the form parameter $H = \delta^*/\theta$ cannot be calculated in the first approximation to the terms of the order of $O(\varepsilon)$ (i.e., $H \equiv 1$) [10] is wrong.

Using (8), we write, analogously to (16),

$$\theta = \int_0^\delta \left(\frac{u}{u_0} - \varepsilon F\right)\left(1 - \frac{u}{u_0} + \varepsilon F\right) dy , \tag{30}$$

then, taking into account (8), we obtain

$$\delta^* = -\,\varepsilon\,[b\,(x) + \varepsilon c\,(x) + O\,(\varepsilon^2)]\,, \quad \theta = -\,\varepsilon\,[b\,(x) + \varepsilon c\,(x) + \varepsilon \int_0^\delta f^2 dy + O\,(\varepsilon^2\delta)]\,, \tag{31}$$

$$b\,(x) = \int_0^\delta f dy\,, \quad c\,(x) = \int_0^\delta E dy\,.$$

Hence, for the form parameter $H = \delta^*/\theta$ we have

$$H = [1 - \varepsilon I_2/I_1]^{-1} + O\,(\varepsilon^2)\,, \quad I_1 = -\int_0^\infty f d\zeta = 1\,, \quad I_2 = \int_0^\infty f^2 d\zeta\,. \tag{32}$$

The experiments of F. R. Kham, the results of which are presented in [15], have shown that the data of calculation by (32) agree well with the experimental data for smooth and rough plates at ε = 0.035–0.1 if $I_2/I_1 \approx 6.3$.

Possibility of Application of the Results Obtained to Real Regimes of Flow in a Turbulent Boundary Layer. The results obtained in the present work allow the following main conclusions: the disturbances of the velocity in a turbulent boundary layer should be separated into vortex and potential disturbances and the velocity profiles (22) are locally self-similar. The first of these conclusions is indirectly supported by the results obtained in [11], where the author explains the failure in the calculation of the Reynolds stresses by the influence of $\partial p/\partial y$. An asymptotic analysis performed in [6] has also shown theoretically that it is necessary to take into account the transverse pressure gradient. However, it is necessary to perform direct experiments in which the vortex and potential disturbances will be separated. To prove the rightfulness of this division, Yu. A. Lashkov, V. M. Litvinov, N. V. Samoilova, and A. A. Uspenskii have measured the velocity profiles on the plane surface of the working part of a wind tunnel in the preseparation zone of the flow downstream of the obstacle. It should be noted that these measurements were made, in the strict sense, not in the region of the boundary layer where the longitudinal pressure gradient is specified by the solution of the Euler equations ("inviscid" flow) and the differential head normal to the body is determined by the surface curvature (relation (9)). In the preseparation zone, the transverse pressure gradient arises because of the significant (variable in y) curvature of the streamlines, and the potential component of the velocity disturbances is not described by relation (11).

However, the vortex component of the velocity can be separated in this regime too. To do this, it is necessary to perform measurements without regard for the differential head across the boundary layer, assuming that the mutual influence of the vortex and potential disturbances can be ignored (the linear approximation is true). This corresponds to the "standard" method of determining the average velocity by the difference between the total pressure measured using a corresponding head and the static pressure on the surface of the body. The true average velocity can be sought with the use of a thermoanemometric sensor or (as in the indicated experiments) by measuring the static pressure not on the surface of the body but at the value of y at which the total pressure was measured. The results of the corresponding measurements are shown in Fig. 1 (curves 1 and 2). Here, curve 3, obtained by subtraction of the potential disturbances Δu_0 from the total velocity, is presented for estimation of the error of the linear approximation (i.e., the independence of the vortex and potential disturbances). The potential disturbances have been calculated by linearization

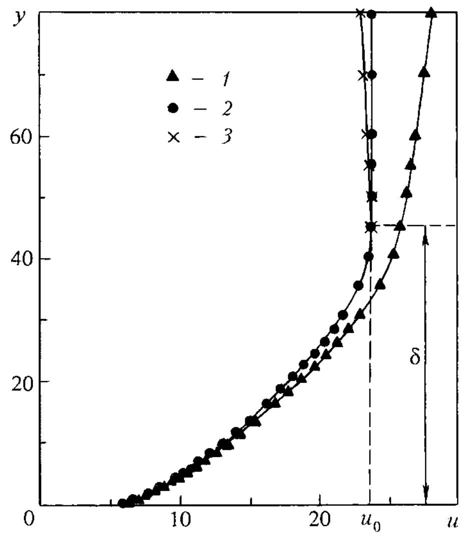

Fig. 1. Profiles of the tangential velocity in the preseparation region of the boundary layer: 1) total velocity, 2) velocity from which the potential component is excluded in the process of measurements, 3) velocity from which the potential component calculated in the linear approximation is excluded. *y*, mm; *u*, m/sec.

of the Bernoulli equation near the value of u_0 by the difference between the static pressures measured at $y = 0$ and at the current value of *y*.

The results presented in Fig. 1 point to the fact that the upper boundary of a boundary layer cannot be determined without separation of the vortex disturbances in the flows having a large curvature of their lines. It also follows from this figure that, in the general case, the velocity u_0 cannot be identified with the velocity at the outer boundary of the layer. Moreover, the coincidence of curves 2 and 3 in the region of the boundary layer testifies to the fact that the vortex and potential disturbances are fairly small and therefore develop practically independently.

The above conclusion that the profiles of the vortex component of the velocity-defect profile are locally self-similar (in the main asymptotic approximation) should be verified more extensively in experiments for the cases of real regimes of flow. For example, numerous experimental data obtained at $\mathrm{Re} \approx 10^6$–10^7 in [16] show that a fairly large "relaxation length" (approaching zero at $\varepsilon \to 0$) is required in order that a quasiequilibrium regime of flow can be established ($\Pi = \Pi(\beta)$). "Relaxation" equations have been proposed in [1] for taking into account the "delay" in establishment of equilibrium. However, they cannot be considered as sufficiently valid, in particular because of the fact that the "relaxation length" was taken to be proportional to δ^* and not to the thickness of the boundary layer. Nonetheless, an acceptable relaxation equation can be derived by processing a large number of experimental data.

This approach makes it possible to obtain a method for calculating a turbulent boundary layer without recourse to fairly arbitrary models of turbulent viscosity.

Conclusions. The asymptotic analysis performed by us has shown that a flow in a turbulent boundary layer differs substantially from an analogous flow in a laminar boundary layer. It has been established that models of turbulent viscosity, leading to parabolic equations similar to the laminar-layer equations, are in contradiction with the main (convective) mechanism of momentum transfer caused by the coherent vortex structures.

The second feature of a flow in a turbulent layer is that the influence of the transverse pressure gradient should be taken into account in the cases of a large curvature of the streamlines. Because of this, in measurements of the boundary-layer parameters the potential disturbance caused by the transverse pressure gradient should be excluded from the total velocity.

Unlike the case of a laminar boundary layer where real regimes of flow are described very well in the asymptotic approximation, in the case of a turbulent layer an analogous approximation gives a quantitatively acceptable solution only for unreally large values of $\mathrm{Re} = O(10^{12})$ because of the logarithmic dependence of the small parameter ε on Re. However, many asymptotic properties of the turbulent boundary layer are retained up to $\mathrm{Re} = O(10^6)$.

These properties make it possible to develop calculation methods that do not involve the use of turbulent-viscosity models.

NOTATION

A, constant inverse to the universal von Kármán constant; a, parameter determining the form of the logarithmic variety of the function f (13); b, c, coefficients of the asymptotic expansion of δ^* (31); C, constant specifying the initial value of the function γ (19); c_f, local friction coefficient (1); D, constant (21); E, F, f, functions in the corresponding terms of the asymptotic expansion of u/u_0 (8); $H = \delta^*/\theta$, form parameter (32); h, characteristic transverse dimension of the near-wall zone of the flow (25); I_1, I_2, values of intervals (32); k_e, effective hydraulic roughness; k_e^+, k_e related to the "viscous length $\nu/(\varepsilon u_0)$"; m, constant (21); u = 1, 2, 4, ..., integer; p, pressure; p_w, pressure on the surface of the body; r, local radius of the surface curvature; R, characteristic transverse dimension of the channel; Re, Reynolds number; $\mathrm{Re}_x = u_0\delta/\nu$ (27); $\mathrm{Re}_* = u_0 x/\nu$ (28); T, dimensionless shear stress (12); u_0, velocity calculated using the Bernoulli equation at $p = p_w$; u and v, velocity components along the surface of the body and on the normal to it; x and y, coordinates along the surface of the body and on the normal to it; α, ratio of k_e to the average geometric roughness; β, Clauser parameter (18); $\gamma = \delta^*/\varepsilon^2$ (5); δ, transverse dimension of the boundary layer; δ^*, displacement thickness of the boundary layer (16); $\varepsilon = (c_f/2)^{1/2}$ (1); $\eta = y/\varepsilon$ (8); θ, momentum thickness of the boundary layer (30); λ, friction coefficient of the channel [3]; ν, kinematic viscosity; Π, Coles parameter of the wake in the turbulent boundary layer [11]; ρ, density; τ_{xx}, τ_{xy}, τ_{yy}, components of the Reynolds stress tensor; τ_w, shear stress of the surface of the body; φ, function of the first term in the asymptotic expansion of v/u_0 (8); ω (α), empirical function [3]. Subscripts: e, effective; f, friction; 0, boundary layer is absent; w, wall.

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